Some explicit travelling-wave solutions of a perturbed sine-Gordon equation

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Abstract

We present in closed form some special travelling-wave solutions (on the real line or on the circle) of a perturbed sine-Gordon equation. The perturbation of the equation consists of a constant forcing term γ and a linear dissipative term, and the equation is used to describe the Josephson effect in the theory of superconductors and other remarkable physical phenomena. We determine all travelling-wave solutions with unit velocity (in dimensionless units). For $|\gamma| \leq 1$ we find families of solutions that are all (except the obvious constant one) manifestly unstable, whereas for $|\gamma| > 1$ we find families of stable solutions describing each an array of evenly spaced kinks.

Preprint 07-09 Dip. Matematica e Applicazioni, Università di Napoli. To appear in the proceedings of the conference "Mathematical Physics Models and Engineering Sciences", Naples 22-23 June 2006, in honour of P. Renno's for his 70^{th} birthday.

1 Introduction and preliminaries

The scope of this communication is the determination in closed form of some special solutions of the class of partial differential equations

$$\varphi_{tt} - \varphi_{xx} + \sin\varphi + \alpha\varphi_t + \gamma = 0 \qquad x \in \mathbb{R}, \tag{1}$$

parametrized by constants $\alpha > 0, \gamma \in \mathbb{R}$, more precisely the determination of the travelling-wave solutions $\varphi(x,t) = \tilde{g}(x-vt)$ with velocity $v = \pm 1$.

This equation (here written in dimensionless units) has been used to describe with a good approximation a number of interesting physical phenomena, notably Josephson effect in the theory of superconductors [6], which is at the base [1] of a large number of advanced developments both in fundamental research (e.g. macroscopic effects of quantum physics, quantum computation) and in applications to electronic devices (see e.g. Chapters 3-6 in [2]), or more recently also the propagation of localized magnetohydrodynamic modes in plasma physics [9]. The last two terms are respectively a dissipative and a forcing one; the sine-Gordon equation (sGe) is obtained by setting them equal to zero.

The sGe describes also the dynamics of the continuum limit of a sequence of neighbouring heavy pendula constrained to rotate around the same horizontal x-axis and coupled to each other through a torque spring [8] (see fig. 1); $\varphi(x,t)$ is the deviation angle from the lower vertical position at time t of the pendulum having position x. One can model also the dissipative term $-\alpha\varphi_t$ of (1) by immersing the pendula in a linearly viscous fluid, and the forcing term γ by assuming that a uniform, constant torque distribution is applied to the pendula. This mechanical analog allows a qualitative comprehension of the main features of the solutions, e.g. of their instabilities. The constant solutions of (1) exist only for $|\gamma| \le 1$ and are, mod 2π,

$$\varphi^s(x,t) \equiv -\sin^{-1}\gamma, \qquad \varphi^u(x,t) \equiv \sin^{-1}\gamma + \pi. \tag{2}$$

If $|\gamma| < 1$ the former is stable, the latter unstable, as they yield respectively local minima and maxima of the energy density

$$h := \frac{\varphi_t^2}{2} + \frac{\varphi_x^2}{2} + \gamma\varphi - \cos\varphi. \tag{3}$$

In the mechanical analog they respectively correspond to configurations with all pendula hanging down or standing up. If $\gamma = \pm 1$ $\varphi^s = \varphi^u = \mp\pi/2$ (mod 2π), which is unstable because it is an inflection point for h.

In [5] we have performed a detailed analysis of travelling-wave solutions of (1). We briefly recall the framework adopted there and some of the results. Without loss of generality we can and shall assume $\gamma \ge 0$: if originally $\gamma < 0$, we just need to replace $\varphi \to -\varphi$. Moreover, space or time translations transform any solution into a two-parameter family of solutions, so one can choose any of them as the family representative element; for travelling-wave solutions this reduces to translation of the only independent variable. In agreement with the conventions adopted in [5], we specify our travelling-wave Ansatz as follows:

$$\xi := \pm x - t, \qquad \varphi(x,t) = g(\xi) - \pi. \tag{4}$$

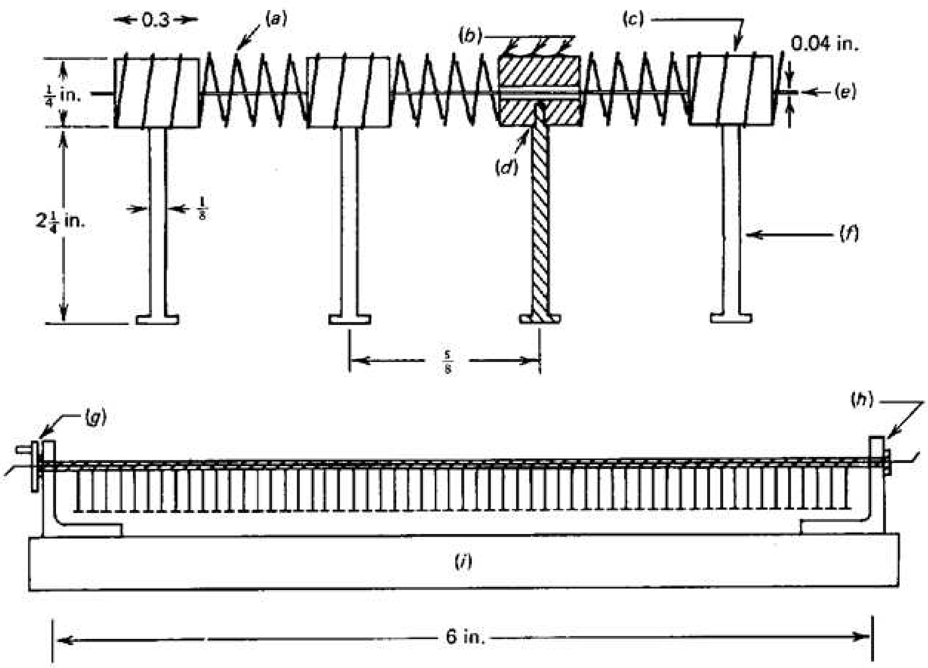

Figure 1: Mechanical model for the sine-Gordon equation. (a) Spring, (b) solder, (c) brass, (d) tap and thread, (e) wire, (f) nail, (g) and (h) ball bearings, (i) base (After A. C. Scott [8], courtesy of A. Barone, see [1])

Replacing the Ansatz in (1) one obtains the first order ordinary differential equation

$$\alpha g' = \gamma - \sin g. \tag{5}$$

We have already recalled the constant solutions. If g' is not identically zero, by integrating $d\xi = \alpha dg/(\gamma - \sin g)$ one finds

$$\xi - \xi_0 = \int_{\xi_0}^{\xi} d\xi = \alpha \int_{g_0}^{g} \frac{ds}{\gamma - \sin s}$$

separately in each interval in which g' keeps its sign. This allows to determine the solution implicitly, namely the inverse $\xi(g)$.

If $\gamma \leq 1$, as g approaches respectively $\sin^{-1}\gamma$ or $\pi - \sin^{-1}\gamma$ (mod. 2π) the denominator of the integrand goes to zero (linearly if $\gamma < 1$, quadratically if $\gamma = 1$) while keeping the same sign, and therefore the integral diverges, implying that the corresponding ξ respectively goes either to $\pm\infty$, or to $\mp\infty$ [5]. In either case the range of $\xi(g)$ is the whole $\mathbb{R}$, implying that by taking the inverse one obtains $g(\xi)$ already in all the domain. If $\gamma > 1$ the denominator of the integrand is positive for all $s \in \mathbb{R}$, so that the solution g is strictly monotonic and linear-periodic, i.e. the

sum of a linear and a periodic function, and

$$g(\xi+\Xi)=g(\xi)+2\pi, \qquad\qquad \Xi:=\alpha\int_0^{2\pi}\frac{ds}{\gamma-\sin s}. \tag{6}$$

Denoting as $\check{\varphi}^{\pm}$ the corresponding solutions with $\xi:=\pm x-t$, by (4) this implies

$$\check{\varphi}^{\pm}(x+\Xi,t)=\check{\varphi}^{\pm}(x,t)\pm 2\pi. \tag{7}$$

This behaviour is illustrated in fig. 2 by a picture of the corresponding configuration for the mechanical model of fig. 1.

$\check{\varphi}^{\pm}$ can be interpreted also as solutions of (1) on a circle of length $L=m\Xi$, for any $m\in\mathbb{N}$. The integer m parameterizes different topological sectors: in the m-th the pendula chain twists around the circle m times.

2 Explicit travelling-wave solutions with unit velocity

The purpose of this work is to determine in closed form the travelling-wave solutions (4) just described. We first transform eq. (5), with the help of the identities (18), into

$$4\alpha\frac{F'}{1+F^2}=\gamma-4\frac{F(1-F^2)}{(1+F^2)^2}$$

by looking for g in the form $g=4\tan^{-1}F$ and then into

$$2\alpha y'=2y+\gamma(1+y^2) \tag{8}$$

by looking for F in the form $F=y+\sqrt{1+y^2}$. Note that diverging of $|y|$ at some point ξ_0 does not affect the continuity (and smoothness) of g at ξ_0, even if the right limit is ∞ and the left one is $-\infty$, or viceversa: $y\to\pm\infty$ respectively implies $F\to\infty,0$ whence $g\to 0$ mod 2π in either case, which is compatible with a continuous g.

Below we solve for $y(\xi)$ explicitly. Putting all redefinitions together, we shall find solutions φ through the formula

$$\varphi^{\pm}(x,t)=4\tan^{-1}\left[y(\pm x-t)+\sqrt{1+y^2(\pm x-t)}\right]-\pi. \tag{9}$$

Only if $\gamma\le 1$ the solutions $y_\pm=-\gamma^{-1}\pm\sqrt{\gamma^{-2}-1}$ of the second degree equation $y^2+y2/\gamma+1=0$ are real and therefore give (real) constant solutions $y(\xi)\equiv y_\pm$ of (8), whence the already mentioned constant solutions φ^s,φ^u of (1). For nonconstant solutions (8) is equivalent to

$$d\xi=\frac{2\alpha}{\gamma}\frac{dy}{y^2+\frac{2}{\gamma}y+1} \tag{10}$$

separately in each interval where y' keeps its sign. The discussion of (10) depends now on the value of the discriminant $\Delta=4/\gamma^2-4$ of the equation $y^2+y2/\gamma+1=0$.

If $\gamma<1$, then $\Delta>0$, $y_\pm$ are real and different and (10) can be written as

$$d\xi=\frac{2\alpha}{\gamma}\frac{dy}{(y-y_+)(y-y_-)}=\frac{\alpha}{\sqrt{1-\gamma^2}}\left[\frac{dy}{y-y_+}-\frac{dy}{y-y_-}\right],$$

which is integrated to give the two families of solutions

$$y_1(\xi)=\frac{y_++y_-e^{A(\xi-\xi_0)}}{1+e^{A(\xi-\xi_0)}},\qquad\qquad y_2(\xi)=\frac{y_+-y_-e^{A(\xi-\xi_0)}}{1-e^{A(\xi-\xi_0)}},\tag{11}$$

where $A:=(\sqrt{1-\gamma^2})\alpha^{-1}$ and ξ_0 is an integration constant. One easily checks that y_1',y_2' (and therefore also g_1',g_2') are respectively negative-, positive-definite; and that $\lim_{\xi\to\pm\infty}y_i(\xi)=y_\mp$ for both $i=1,2$. Using formulae (23-24) shown in the Appendix we thus find

$$\lim_{\xi\to\infty}F[y_i(\xi)]=F(y_-)=\tan\theta,\qquad\qquad \lim_{\xi\to-\infty}F[y_i(\xi)]=F(y_+)=\tan\left(\frac{\pi}{4}-\theta\right).$$

for both $i=1,2$, and mod 2π on one side a stricly decreasing $g_1(\xi)$ with

$$\lim_{\xi\to-\infty}g_1=\pi-\sin^{-1}\gamma,\qquad\qquad \lim_{\xi\to\infty}g_1=\sin^{-1}\gamma,$$

and on the other a stricly increasing $g_2(\xi)$ with

$$\lim_{\xi\to-\infty}g_2=\pi-\sin^{-1}\gamma,\qquad\qquad \lim_{\xi\to\infty}g_2=2\pi+\sin^{-1}\gamma.$$

As already noted, the singularity of y_2 at $\xi=\xi_0$ does not affect the continuity (and smoothness) of g_2. Correspondingly, mod 2π

$$\lim_{x\to\mp\infty}\varphi_1^\pm=-\sin^{-1}\gamma\equiv\varphi^s,\qquad\qquad \lim_{x\to\pm\infty}\varphi_1^\pm=-\pi+\sin^{-1}\gamma\equiv\varphi^u,\tag{12}$$

$$\lim_{x\to\mp\infty}\varphi_2^\pm=-\sin^{-1}\gamma\equiv\varphi^s,\qquad\qquad \lim_{x\to\pm\infty}\varphi_2^\pm=\pi+\sin^{-1}\gamma\equiv\varphi^u,\tag{13}$$

therefore $\varphi_1^\pm,\varphi_2^\pm$ are unstable solutions, as noted in [5].

If $\gamma=1$, then $\Delta=0$, $y_\pm=-1$ and (10) can be written as

$$d\xi=2\alpha\,\frac{dy}{(y+1)^2}=-2\alpha\,d\left[\frac{1}{y+1}\right],$$

which is integrated to give

$$y(\xi)=-\left[1+\frac{2\alpha}{\xi-\xi_0}\right].\tag{14}$$

This implies, with the help of (25),

$$\lim_{\xi\to\pm\infty}y(\xi)=-1,\qquad\Rightarrow\qquad \lim_{\xi\to\pm\infty}F[y(\xi)]=\sqrt{2}-1=\tan\frac{\pi}{8},$$

whereas again the singularity of y at $\xi=\xi_0$ does not affect the continuity of g. As y', and therefore also F', g', are positive-definite, one finds mod 2π

$$\lim_{\xi\to-\infty} g=\frac{\pi}{2}, \qquad\qquad \lim_{\xi\to\infty} g=\frac{5\pi}{2}$$

and, correspondingly,

$$\lim_{x\to\mp\infty} \varphi^{\pm}=-\frac{\pi}{2}, \qquad\qquad \lim_{x\to\pm\infty} \varphi^{\pm}=\frac{3\pi}{2}; \tag{15}$$

also these $\varphi^{\pm}$ are unstable, as noted in [5].

Finally, if $\gamma>1$, then $\Delta<0$, $y_{\pm}$ are complex conjugate and the denominator of (10) does not vanish for any value of y. Setting $w:=(y\gamma+1)/\sqrt{\gamma^2-1}$ (10) can be written as

$$d\xi=\frac{2\alpha}{\sqrt{\gamma^2-1}}\frac{dw}{1+w^2},$$

which is integrated to give $\xi-\xi_0=2\alpha\tan^{-1}w/\sqrt{\gamma^2-1}$, whence

$$y(\xi)=-\gamma^{-1}+\frac{1}{\sqrt{1-\gamma^{-2}}}\tan\left[\frac{\sqrt{\gamma^2-1}}{2\alpha}(\xi-\xi_0)\right], \tag{16}$$

where ξ_0 is an integration constant. This is a periodic function with period

$$\Xi:=\frac{2\pi\alpha}{\sqrt{\gamma^2-1}}, \tag{17}$$

and the latter is also the period occurring in (6). In fact, choosing $\xi_0=0$ for simplicity, we see that as ξ varies from $-\Xi/2$ to $\Xi/2$ $y(\xi)$ varies from $-\infty$ to ∞, $F(\xi)$ varies from 0 to ∞, $g(\xi)$ varies from 0 to 2π. By continuity of g (which again is not affected by the singularity of y at $\xi=\Xi(k+1/2)$ ($k\in\mathbb{Z}$), we thus find the behaviour (6). The corresponding solutions $\check{\varphi}^{\pm}$ fulfill (7), describe arrays of evenly-spaced kinks (see fig. 2) moving with velocity ±1 and are stable [5] (see also [7, 3]).

Appendix

We first recall the trigonometric identities

$$\sin 2\alpha=\frac{2\tan\alpha}{1+\tan^2\alpha}, \quad \cos 2\alpha=\frac{1-\tan^2\alpha}{1+\tan^2\alpha} \qquad\Rightarrow\qquad \sin 4\alpha=\frac{4\tan\alpha(1-\tan^2\alpha)}{(1+\tan^2\alpha)^2}. \tag{18}$$

Given $\gamma\in[0,1]$, let $\theta:=\frac{1}{4}\sin^{-1}\gamma\in[0,\frac{\pi}{8}]$. Then $\gamma=\sin 4\theta$, $\sqrt{1-\gamma^2}=\cos 4\theta$ and, using the bisection formulae,

$$\sqrt{1+\sqrt{1-\gamma^2}}=\sqrt{1+\cos 4\theta}=\sqrt{2}\cos 2\theta, \tag{19}$$

$$\sqrt{1-\sqrt{1-\gamma^2}}=\sqrt{1-\cos 4\theta}=\sqrt{2}\sin 2\theta, \tag{20}$$

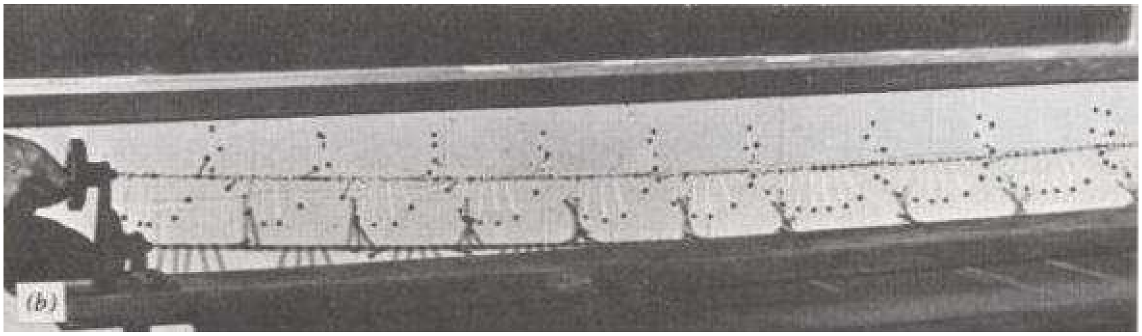

Figure 2: Photographs of the mechanical model of fig. 1: evenly spaced array of kinks (After A. C. Scott [8], courtesy of A. Barone, see [1])

whence in turn

$$\sqrt{2}-\sqrt{1+\sqrt{1-\gamma^2}} = \sqrt{2}(1-\cos 2\theta) = 2\sqrt{2}\sin^2\theta, \tag{21}$$

$$\sqrt{2}-\sqrt{1-\sqrt{1-\gamma^2}} = \sqrt{2}(1-\sin 2\theta) = \sqrt{2}\left[1-\cos\left(\frac{\pi}{2}-2\theta\right)\right] = 2\sqrt{2}\sin^2\left(\frac{\pi}{4}-\theta\right). \tag{22}$$

Hence, using also the sinus duplication formula, we end up with

$$\begin{aligned} F(y_+) &= \frac{\sqrt{1-\sqrt{1-\gamma^2}}}{\gamma}\left[\sqrt{2}-\sqrt{1-\sqrt{1-\gamma^2}}\right] = \frac{4\sin 2\theta\sin^2\left(\frac{\pi}{4}-\theta\right)}{\sin 4\theta} \\ &= \frac{2\sin^2\left(\frac{\pi}{4}-\theta\right)}{\cos 2\theta} = \frac{2\sin^2\left(\frac{\pi}{4}-\theta\right)}{\sin\left(\frac{\pi}{2}-2\theta\right)} = \frac{\sin\left(\frac{\pi}{4}-\theta\right)}{\cos\left(\frac{\pi}{4}-\theta\right)} = \tan\left(\frac{\pi}{4}-\theta\right), \end{aligned} \tag{23}$$

$$F(y_-) = \frac{\sqrt{1+\sqrt{1-\gamma^2}}}{\gamma}\left[\sqrt{2}-\sqrt{1+\sqrt{1-\gamma^2}}\right] = \frac{4\cos 2\theta\sin^2\theta}{\sin 4\theta} = \frac{2\sin^2\theta}{\sin 2\theta} = \tan\theta. \tag{24}$$

If we choose $\gamma = 1$ in (21) and use the sinus duplication formula we find as another consequence

$$\sqrt{2}-1 = 2\sqrt{2}\sin^2\left(\frac{\pi}{8}\right) = \frac{2\sin^2\left(\frac{\pi}{8}\right)}{\sin\left(\frac{\pi}{4}\right)} = \frac{\sin\frac{\pi}{8}}{\cos\frac{\pi}{8}} = \tan\frac{\pi}{8}. \tag{25}$$

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